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An asymptotic approach to compressible boundary-layer flow

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Abstract—A regular perturbation approach is applied to take into account variable property effects. Of special interest is the effect of the pressure-dependent density. In the framework of an asymptotic theory, compressibility effects are considered as variable property effects. By means of this asymptotic theory the deviation of skin friction and heat transfer results from their incompressible, isothermal values are determined for laminar Falkner–Skan boundary layers. As far as laminar flow is concerned there is no need for any empirical information.

1. INTRODUCTION

IN THE theory of compressible boundary layers one should accommodate several complications not present in the incompressible case: density should be considered as a new variable that is pressure and temperature dependent. An equation of state is needed to relate thermodynamic properties. Momentum and energy equations are coupled and nonlinear. Since viscous heating may result in considerable temperature gradients the temperature dependence of transport and thermodynamic properties must be taken into account. All these are far from trivial matters, and hence there are very few exact solutions for compressible viscous flows, see White [1] for a survey of solutions including compressibility effects.

As far as boundary-layer flow is concerned the purely numerical approach is straightforward taking into account the dependence on pressure and temperature of all physical properties involved in the problem. The more analytical approach by so-called compressibility transformations is, for practical purposes, restricted to a narrow class of flows exhibiting some kind of self similarity. The basic objective of these transformations is to derive a set of functions which relate a given compressible flow to a corresponding incompressible flow; see, for example, Stewartson [2]. In other words this method aims at converting a variable property flow to a constant property flow.

In contrast to this, the basic idea behind the present study is to account for variable property effects, i.e. compressibility effects, by a regular perturbation procedure as described, for example, by Van Dyke [3]. The zero-order solution of this asymptotic approach is the constant property incompressible flow. The linear terms of a subsequent perturbation procedure can cover compressibility effects to an extent that makes an extension to higher-order terms unnecessary for many practical applications.

The asymptotic approach to account for compressibility effects is by no means restricted to laminar flow. Nevertheless for turbulent flows its application is complicated considerably by the need for turbulence modelling which itself is affected by variable property effects. That is why only laminar boundary-layer flow is considered in this study. The basic ideas underlying its possible extension to turbulent flows are given in ref. [4].

Within the theory of laminar boundary layers, self-similar solutions are often taken as a starting point for a more general investigation. Besides the fact that they are of practical importance by themselves, like flat plate and stagnation point flow they are often used as a basis for general methods of calculation like integral methods; see, for example, Walz [5]. In the incompressible limit self-similar flows like the well-known Falkner–Skan (wedge-type outer flow) similarity solutions have a clear physical interpretation. The momentum equation decoupled from thermal effects in the incompressible limit after a transformation of the normal coordinate is dependent on one similarity coordinate only. If compressibility and thermal effects are now taken into account an additional transformation of the streamwise coordinate is necessary. As a consequence of this the connection to the inviscid outer flow, given in the original streamwise coordinate, is complicated considerably.

This difficulty can be circumvented by the asymptotic approach applied in this study. The dependence on the streamwise coordinate of compressibility and other variable property effects, which will depend on the thermal boundary conditions, can be taken into account by appropriate streamwise-dependent factors in the perturbation terms without need for a streamwise coordinate transformation.

As mentioned earlier, considerable temperature gradients may result from viscous heating. As a consequence the temperature dependence of all physical properties involved in a problem should be taken into account. Therefore, it is reasonable to treat

NOMENCLATURE

a	velocity of sound
B	constant, equation (6)
c_f	skin friction coefficient
c_p	specific heat at constant pressure
e	exponent: $e = 0$ for $T_w = \text{const}$; $e = (1-m)/2$ for $q_w = \text{const}$
Ec	Eckert number, $u_o^{*2}/c_p^* T^*$
f	dimensionless streamfunction
h	enthalpy
H	total enthalpy
K_α	property of the fluid, equation (5)
$\bar{K}_\rho$	property of the density, equation (4b)
L	reference length
m	exponent, equation (6)
m_α	exponent, equation (19)
Ma	Mach number, u_o^*/a^*
p	pressure
P	pressure coefficient, equation (4a)
Pr	Prandtl number, $\eta^* c_p^*/\lambda^*$
q	heat flux
r	recovery factor, equation (25)
Re	Reynolds number, $\rho^* u_o^* L^*/\eta^*$
T	temperature
T_r	reference temperature
u	velocity
x, y	coordinates.

Greek symbols

α	physical property
β	wedge angle, Fig. 1
γ	ratio of heat capacities
ε	heat transfer parameter, equation (2)
$\bar{\varepsilon}$	high speed parameter, equation (1a)
η	viscosity
η_s	similarity variable, equation (7)
θ	dimensionless temperature
λ	thermal conductivity
ψ	streamfunction
ρ	density
τ	shear stress.

Subscripts

α	associated with the property $\alpha(\rho, \eta, \ldots)$
aw	adiabatic wall
cp	constant property
e	outer edge of the boundary
w	wall
o	reference condition
I	associated with heat transfer effects
II	associated with high speed effects.

the problem by a double perturbation procedure with one perturbation parameter for high speed (compressibility) effects and one for heat transfer effects.

The influence of the heat transfer parameter alone has been investigated in the past in several studies; see, for example Carey and Mollendorf [6] for natural convection flows, and an extensive study by the present author [7].

The extension to compressible flows is the main object of this study. The procedure is demonstrated for laminar Falkner–Skan-like boundary layers.

2. PERTURBATION PARAMETERS

Since physical properties are pressure and temperature dependent, two perturbation parameters are necessary to cover all variable property effects by a perturbation procedure.

One (small) perturbation parameter, from now on called the '*high speed parameter*' is

$$\bar{\varepsilon} = \frac{u_o^{*2}}{c_{po}^* T_o^*}. \tag{1a}$$

Starred quantities are dimensional throughout this study, the subscript 'o' refers to a reference state fixed later. This parameter $\bar{\varepsilon}$ is the well-known Eckert number Ec, an important parameter in connection with viscous heating and pressure work. For perfect gases it is related to the Mach number through

$$\bar{\varepsilon} = Ec_o = (\gamma - 1) Ma_o^2. \tag{1b}$$

The other (small) perturbation parameter, called the '*heat transfer parameter*' is

$$\varepsilon = \frac{T_w^* - T_o^*}{T_o^*} \quad \text{for} \quad T_w = \text{const} \tag{2a}$$

$$\varepsilon = C_1 \frac{q_w^* L^*}{\lambda_o^* T_o^*} \quad \text{for} \quad q_w = \text{const.} \tag{2b}$$

In equation (2b) a constant C_1 is introduced for convenience with $C_1 = [Re_o(m+1)/2]^{-1/2}$, for m see equation (6) below.

3. EXPANSION OF PHYSICAL PROPERTIES

The zero-order solution is the low speed (incompressible) flow with vanishing heat transfer, i.e. the constant property flow. Small deviations from this flow are given by the linear perturbation terms in $\bar{\varepsilon}$ and ε. As far as the physical properties are concerned this is to take into account the linear terms of a Taylor series expansion at the reference state T_o, p_o. For the density it reads:

$$\rho = \frac{\rho^*}{\rho_o^*} = 1+\varepsilon K_\rho \theta_{\rm I} + \bar{\varepsilon}(K_\rho \theta_{\rm II} + \bar{K}_\rho P) + O(\varepsilon^2, \varepsilon\bar{\varepsilon}, \bar{\varepsilon}^2) \qquad (3)$$

with

$$T \equiv (T^* - T_o^*)/T_o^* = T_{\rm I} + T_{\rm II} = \varepsilon\theta_{\rm I} + \bar{\varepsilon}\theta_{\rm II}$$
$$P \equiv (p^* - p_o^*)/\rho^* u_o^{*2} \qquad (4a)$$

and

$$K_\rho = \left(\frac{T^*}{\rho^*}\frac{\partial \rho^*}{\partial T^*}\right)_o; \quad \bar{K}_\rho = \left(c_p^* T^* \frac{\partial \rho^*}{\partial p^*}\right)_o \qquad (4b)$$

being dimensionless properties of the fluid. The dimensionless temperature T is split into two parts: $T_{\rm I} = \varepsilon\theta_{\rm I}$ covering direct heat transfer effects; and $T_{\rm II} = \bar{\varepsilon}\theta_{\rm II}$ associated with the high speed parameter $\bar{\varepsilon}$ covering compressibility effects like viscous heating. According to the two definitions of ε, equations (2a) and (2b), the temperature $\theta_{\rm I}$ is nondimensionalized by $\theta_{\rm I} = (T_{\rm I}^* - T_o^*)/(T_w^* - T_o^*)$ for $T_w = \text{const}$ and $\theta_{\rm I} = (T_{\rm I}^* - T_o^*)/(C_1 q_w^* L^*/\lambda_o^*)$ for $q_w = \text{const}$.

The nondimensional pressure is $P = (p^* - p_o^*)/\rho^* u_o^{*2}$ as usually introduced in low speed flows.

If all properties involved in the problem are now expanded according to equation (3) dimensionless K-numbers like $K_\eta, \bar{K}_\eta, K_\lambda, \ldots$ appear. But since the pressure dependencies of the viscosity η, the thermal conductivity λ and the specific heat capacity c_p are extremely small they are neglected throughout this study. It would be no problem at all to take them into account. The only reason for neglecting all effects in connection with $\bar{K}_\eta$, $\bar{K}_\lambda$ and $\bar{K}_c$ is that they are of no practical interest. The series expansion for a general property α ($\alpha \mathrel{\hat{=}} \eta, \lambda, c_p$) thus is:

$$\alpha = 1 + \varepsilon K_\alpha \theta_{\rm I} + \bar{\varepsilon} K_\alpha \theta_{\rm II} + O(\varepsilon^2, \varepsilon\bar{\varepsilon}, \bar{\varepsilon}^2)$$
$$K_\alpha = \left(\frac{T^*}{\alpha^*}\frac{\partial \alpha^*}{\partial T^*}\right)_o. \qquad (5)$$

If physical properties appear in a fixed combination like $\rho\eta$ they are treated like one single property with $K_{\rho\eta} = K_\rho + K_\eta$.

4. BASIC EQUATIONS AND SOLUTION PROCEDURE

In this study the laminar boundary layer over a wedge will be considered. The outer flow is given by (e $\mathrel{\hat{=}}$ outer edge):

$$u_e^* = B^* x^{*m} \qquad (6)$$

with the exponent m related to the wedge angle β by $m = \beta/(2-\beta)$, see Fig. 1.

Prior to writing down the basic equations the reference quantities must be fixed. Though self-similar flows have no characteristic geometrical length the equations are (formally) nondimensionalized by a length L^*. Temperature and pressure are referred to their stagnation quantities T_o^* and p_o^*, the characteristic values for compressible flows. But instead of choosing $\sqrt{2c_{po}^* T_o^*}$ as the reference velocity a characteristic quantity of the wedge-type outer flow is preferred. The reference velocity u_o^* is the outer flow at a distance L^* from $x^* = 0$ so that $u_e = x^m$ and the quantity B^* drops out of the problem, see equation (6).

With the similarity variable η_s

$$\eta_s = \sqrt{\frac{m+1}{2}}\, x^{(m-1)/2} \int_0^y \rho\, dy \qquad (7)$$

the dimensionless streamfunction $f(x, \eta_s)$

$$f = \frac{\psi^* Re_o^{1/2}}{\rho_o^* u_o^* L^*} \sqrt{\frac{m+1}{2}}\, x^{-(m+1)/2} \qquad (8)$$

and the dimensionless variables according to Table 1, the basic equations read:

$$[\rho\eta f'']' + ff'' + \beta[\rho_e/\rho - f'^2] = \frac{2x}{m+1}\left[f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right] \qquad (9)$$

$$\frac{1}{Pr_o}\left[\frac{\rho\lambda}{c_p}H'\right]' + fH' = \frac{2x}{m+1}\left[f'\frac{\partial H}{\partial x} - H'\frac{\partial f}{\partial x}\right] - \bar{\varepsilon}x^{2m}\left[\rho f' f''\left(\eta - \frac{\lambda}{c_p Pr_o}\right)\right]'. \qquad (10)$$

The associated boundary conditions are:

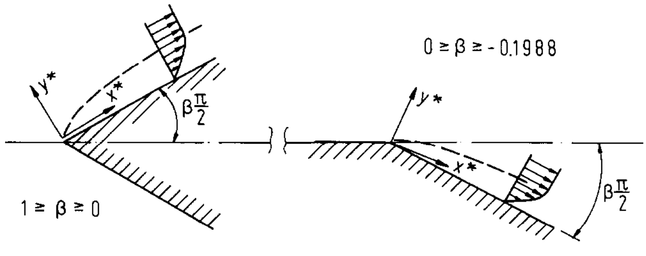

FIG. 1. Wedge-type outer flow.

Table 1. Dimensionless quantities

x	y	u	v	h	H
$\dfrac{x^*}{L^*}$	$\dfrac{y^*}{L^*}Re_o^{1/2}$	$\dfrac{u^*}{u_o^*}$	$\dfrac{v^*}{u_o^*}Re_o^{1/2}$	$\dfrac{h^*-h_o^*}{c_{po}^* T_o^*}$	$\dfrac{H^*-H_o^*}{c_{po}^* T_o^*}$

$$\eta_s = 0:\quad f = f' = 0;\quad H = H_w \text{ for } T_w = \text{const} \qquad H' = H'_w \text{ for } q_w = \text{const} \tag{11a}$$

$$\eta_s \to \infty:\quad f' - 1 = H = 0. \tag{11b}$$

The energy equation is written in terms of the total enthalpy $H(x, \eta_s) = h + \bar{\varepsilon}u^2/2$. $H^* = H_o^*$ is constant in the inviscid outer flow so that the boundary condition is $H = 0$ for $\eta_s \to \infty$. In the following H will be split into two parts, H_I and H_{II}, according to θ_I and θ_{II} [see equation (4a)].

Equations (9) and (10) are a set of nonlinear coupled partial differential equations. If the physical properties are replaced by the Taylor series expansions, equations (3) and (5), there are all together nine parameters in the equations, namely: $\varepsilon, \bar{\varepsilon}, K_\rho, \bar{K}_\rho, K_{\rho\eta}, K_{\rho\lambda}, K_c, Pr, m$. By an appropriate *ansatz* for the dependent variables the number of parameters for the solution procedure can be reduced drastically. With the following *ansatz* seven parameters are separated and only m [velocity exponent, see equation (6)] and Pr (Prandtl number) are left over:

$$f = f_o + \varepsilon x^e[K_{\rho\eta} f_{1I1} + K_\rho f_{1I2}] + \bar{\varepsilon}x^{2m}[K_{\rho\eta} f_{1II1} + K_\rho f_{1II2} + \bar{K}_\rho \bar{f}_1] + O(\varepsilon^2, \varepsilon\bar{\varepsilon}, \bar{\varepsilon}^2) \tag{12}$$

$$H_I = x^e H_{oI} + \varepsilon x^{2e}[K_{\rho\eta} H_{1I1} + K_\rho H_{1I2} + K_{\rho\lambda} H_{1I3} + K_c H_{1I4}] + O(\varepsilon^2) \tag{13}$$

$$H_{II} = x^{2m} H_{oII} + \varepsilon x^{e+2m}[K_{\rho\eta} H_{1II1} + K_\rho H_{1II2} + K_{\rho\lambda} H_{1II3} + K_c H_{1II4} + \bar{K}_\rho \bar{H}_{1II1}] + \bar{\varepsilon}x^{4m}[K_{\rho\eta} H_{1II5} + K_\rho H_{1II6} + K_{\rho\lambda} H_{1II7} + K_c H_{1II8} + \bar{K}_\rho \bar{H}_{1II2}] + O(\varepsilon^2, \varepsilon\bar{\varepsilon}, \bar{\varepsilon}^2). \tag{14}$$

The exponent e is introduced to distinguish between the two kinds of thermal boundary conditions, namely $e = 0$ for $T_w = \text{const}$ and $e = (1-m)/2$ for $q_w = \text{const}$. This parameter e may be looked upon as the 10th parameter of the problem [or the third parameter of the solution procedure with equations (12)–(14)].

In equations (12)–(14) all functions f_i, H_{Ii} and H_{IIi} depend on η_s alone, i.e. they are solutions of ordinary differential equations, typical for self-similar solutions. Since f, H_I and H_{II} include powers of x they are called '*quasi-self-similar*'. The complete set of 22 ordinary differential equations for f_i, H_{Ii} and H_{IIi} is derived in ref. [4]. They are also listed in the Appendix.

It should be emphasized that the original equations (9) and (10) were partial, coupled and nonlinear but that the equations for f_i, H_{Ii} and H_{IIi} are ordinary, uncoupled (in the sense that they can be solved sequentially) and linear (except for the equation for the zero-order streamfunction).

In addition to equations (9)–(14) and the property expansions equations (3) and (5) a relation between total enthalpy and temperature is required to close the set of equations. This relation can be expressed asymptotically, i.e. to the zero and first order, respectively, taking into account that $H = h + \bar{\varepsilon}u^2/2$ and $dh = c_p\,dT + [\rho^{-1} - T(\partial\rho^{-1}/\partial T)_p]\,dp$; see, for example, Baehr [8]. But since the term expressing the pressure dependence of the enthalpy h is extremely small compared to the term $c_p\,dT$ it is neglected throughout this study. It is the second non-asymptotic approximation in this study. Again, like neglecting the pressure dependence of η, λ and c_p, the motivation for this approximation is that it is of no practical interest to keep it asymptotically complete, although this could be done. As far as a perfect gas is concerned it is no approximation at all since $\rho^{-1} - T(\partial\rho^{-1}/\partial T) \equiv 0$ and $dh = c_p\,dT$ is exact. Inserting the asymptotic expansions for all variables one obtains simple relations between H_{iI}, H_{iII} and θ_{iI}, θ_{iII}, respectively. The zero-order relations are:

$$\theta_{oI} = H_{oI} \quad\text{and}\quad \theta_{oII} = H_{oII} - f_o'^2/2. \tag{15}$$

The first-order relations are listed in the Appendix. The numerical solution of the equations for f_i, H_{iI} and H_{iII} is straightforward. In Table 2 results are given for certain values of the parameters m and Pr (which may be used for interpolation if different values are needed) for the thermal boundary conditions $T_w = \text{const}$ and $q_w = \text{const}$.

5. SKIN FRICTION AND HEAT TRANSFER RESULTS

The skin friction under the influence of variable properties related to its constant property value is:

$$\frac{c_f}{c_{fcp}} = (\rho\eta)_w \frac{f_w''}{f_{ow}''} = 1 + \varepsilon x^e\left[K_{\rho\eta}\left(\theta_{oI} + \frac{f_{1I1}''}{f_o''}\right) + K_\rho \frac{f_{1I2}''}{f_o''}\right]_w + \bar{\varepsilon}x^{2m}\left[K_{\rho\eta}\left(\theta_{oII} + \frac{f_{1II1}''}{f_o''}\right) + K_\rho \frac{f_{1II2}''}{f_o''} + \bar{K}_\rho \frac{\bar{f}_1'' - \frac{1}{2}}{f_o''}\right]_w + O(\varepsilon^2, \varepsilon\bar{\varepsilon}, \bar{\varepsilon}^2). \tag{16}$$

Table 2(a). Numerical results; T_w = const

$m =$	0.0		1/3		2/3		1.0	
$Pr =$	0.7	7.0	0.7	7.0	0.7	7.0	0.7	7.0
f''_o	0.4696	0.4696	0.9277	0.9277	1.1203	1.1203	1.2326	1.2326
θ'_{oI}	−0.4139	−0.9135	−0.4705	−1.0930	−0.4873	−1.1491	−0.4959	−1.1784
θ'_{oII}	−0.0340	0.6975	−0.0548	1.1597	−0.0643	1.3684	−0.0700	1.4921
f''_{1I1}	−0.3447	−0.4155	−0.5884	−0.7958	−0.6897	−0.8034	−0.7484	−0.8907
f''_{1I2}	—	—	−0.3472	−0.2737	−0.4875	−0.2957	−0.5691	−0.3515
f''_{1II1}	−0.0370	−0.0326	−0.0564	−0.0294	−0.0648	−0.0299	−0.0699	−0.0305
f''_{1II2}	—	—	−0.1673	−0.1800	−0.2207	−0.2394	−0.2502	−0.2724
$\bar{f}''_1$	0.1174	0.1174	0.2095	0.2095	0.2523	0.2523	0.2775	0.2775
θ'_{1I1}	0.0612	0.1859	0.0537	0.1837	0.0513	0.1831	0.0501	0.1825
θ'_{1I2}	—	—	0.0353	0.0503	0.0437	0.0643	0.0476	0.0716
θ'_{1I3}	0.2496	0.5449	0.2896	0.6590	0.3018	0.6950	0.3080	0.7138
θ'_{1I4}	−0.0427	−0.0881	−0.0544	−0.1125	−0.0581	−0.1204	−0.0601	−0.1246
θ'_{1II1}	0.0512	−0.0010	0.0989	0.0961	0.1221	0.1572	0.1361	0.1974
θ'_{1II2}	—	—	0.0331	−0.0801	0.0444	−0.1563	0.0510	−0.2101
θ'_{1II3}	−0.0140	−0.6673	0.0028	−1.1282	0.0061	−1.3401	−0.0124	−1.4668
θ'_{1II4}	0.0480	−0.0302	0.0520	−0.0315	0.0582	−0.0283	0.0824	−0.0233
θ'_{1II5}	−0.0153	−0.1374	−0.0233	−0.1724	−0.0267	−0.1888	−0.0287	−0.1992
θ'_{1II6}	—	—	0.0061	−0.1759	0.0085	−0.2428	0.0098	−0.2804
θ'_{1II7}	0.0202	0.0259	0.0311	0.0301	0.0357	0.0321	0.0384	0.0335
θ'_{1II8}	−0.0202	−0.0259	−0.0311	−0.0301	−0.0357	−0.0321	−0.0384	−0.0335
$\bar{\theta}'_{1II1}$	−0.1035	−0.2284	−0.0850	−0.2031	−0.0781	−0.1933	−0.0746	−0.1882
$\bar{\theta}'_{1II2}$	−0.0085	0.1744	−0.0117	0.2501	−0.0135	0.2915	−0.0147	0.3167

Table 2(b). Numerical results; q_w = const

$m =$	0.0		1/3		2/3		1.0	
$Pr =$	0.7	7.0	0.7	7.0	0.7	7.0	0.7	7.0
f''_o	0.4969	0.4969	0.9277	0.9277	1.1203	1.1203	1.2326	1.2326
θ_{oI}	1.7421	0.7984	1.7208	0.7505	1.8577	0.7933	2.0167	0.8486
θ_{oII}	−0.0821	0.7636	−0.0822	0.7651	−0.0822	0.7651	−0.0822	0.7652
f''_{1I1}	−0.5619	−0.3253	−1.0170	−0.5372	−1.2877	−0.6589	−1.5093	−0.7559
f''_{1I2}	—	—	−0.5294	−0.1359	−0.8636	−0.2232	−1.1476	−0.2983
f''_{1II1}	−0.0086	−0.3499	−0.0077	−0.5820	−0.0069	−0.6865	−0.0065	−0.7484
f''_{1II2}	—	—	−0.1442	−0.3069	−0.1900	−0.4113	−0.2153	−0.4694
$\bar{f}''_1$	0.1174	0.1174	0.2095	0.2095	0.2523	0.2523	0.2775	0.2775
θ_{1I1}	0.4074	0.1230	0.3412	0.0935	0.3684	0.0999	0.4106	0.1116
θ_{1I2}	—	—	0.1927	0.0227	0.2912	0.0332	0.3907	0.0438
θ_{1I3}	−0.1810	−0.2502	−1.1253	−0.2198	−1.3060	−0.2465	−1.5405	−0.2839
θ_{1I4}	−0.3365	−0.0685	−0.3553	−0.0618	−0.4195	−0.0682	−0.4930	−0.0762
θ_{1II1}	0.1236	0.2346	0.1947	0.2242	0.2440	0.2518	0.2877	0.2820
θ_{1II2}	—	—	0.0669	0.0233	0.0812	−0.0002	0.0933	−0.0239
θ_{1II3}	0.0477	−0.4507	−0.0076	−0.4273	−0.0425	−0.4520	−0.0728	−0.4883
θ_{1II4}	0.0953	−0.1590	0.1498	−0.1469	0.1952	−0.1550	0.2386	−0.1610
θ_{1II5}	−0.0460	−0.0325	−0.0384	0.0276	−0.0369	0.0380	−0.0363	0.0422
θ_{1II6}	—	—	0.0041	−0.0859	0.0048	−0.1015	0.0051	−0.1081
θ_{1II7}	0.0500	−0.1817	0.0414	−0.1893	0.0397	−0.1907	0.0390	−0.1913
θ_{1II8}	−0.0534	−0.1098	−0.0448	−0.1034	−0.0431	−0.1020	−0.0424	−0.1015
$\bar{\theta}_{1II1}$	0.4355	0.1996	0.4030	0.1764	0.4029	0.1733	0.4103	0.1744
$\bar{\theta}_{1II2}$	0.0	0.0	0.0	0.0	0.0	−0.0001	0.0	−0.0001

The heat transfer results are:

For T_w = const in terms of q_w:

$$
\begin{aligned}
-q_w\left[\frac{Re_o}{2}(m+1)x^{m-1}\right]^{-1/2} &= (\rho\lambda)_w T'_w = \varepsilon x^e \theta'_{oIw} + \bar{\varepsilon} x^{2m}\theta'_{oIIw} \\
&+ \varepsilon^2 x^{2e}[K_{\rho\eta}\theta'_{1I1} + K_\rho \theta'_{1I2} \\
&+ K_{\rho\lambda}(\theta'_{oI}\theta_{oI} + \theta'_{1I3}) + K_c \theta'_{1I4}]_w \\
&+ \varepsilon\bar{\varepsilon} x^{e+2m}[K_{\rho\eta}\theta'_{1II1} + K_\rho \theta'_{1II2} \\
&+ K_{\rho\lambda}(\theta'_{oII}\theta_{oI} + \theta'_{1II3}) + K_c \theta'_{1II4} \\
&+ \bar{K}_\rho(\bar{\theta}'_{1II1} - \tfrac{1}{2}\theta'_{oI})]_w \\
&+ \bar{\varepsilon}^2 x^{4m}[K_{\rho\eta}\theta'_{1II5} + K_\rho \theta'_{1II6} + K_{\rho\lambda}\theta'_{1II7} \\
&+ K_c \theta'_{1II8} + \bar{K}_\rho(\bar{\theta}'_{1II2} - \tfrac{1}{2}\theta'_{oII})]_w \\
&+ O(\varepsilon^3, \varepsilon^2\bar{\varepsilon}, \varepsilon\bar{\varepsilon}^2, \bar{\varepsilon}^3). \qquad (17)
\end{aligned}
$$

For $q_w = \text{const}$ in terms of T_w:

$$
\begin{aligned}
T_w = {} & \varepsilon x^e \theta_{\text{oIw}} + \bar{\varepsilon} x^{2m} \theta_{\text{oIIw}} \\
& + \varepsilon^2 x^{2e} [K_{\rho\eta} \theta_{111} + K_\rho \theta_{112} + K_{\rho\lambda} \theta_{113} + K_c \theta_{114}]_w \\
& + \varepsilon\bar{\varepsilon} x^{e+2m} [K_{\rho\eta} \theta_{1\text{II}1} + K_\rho \theta_{1\text{II}2} + K_{\rho\lambda} \theta_{1\text{II}3} \\
& \qquad + K_c \theta_{1\text{II}4} + \bar{K}_\rho \bar{\theta}_{1\text{II}1}]_w \\
& + \bar{\varepsilon}^2 x^{4m} [K_{\rho\eta} \theta_{1\text{II}5} + K_\rho \theta_{1\text{II}6} + K_{\rho\lambda} \theta_{1\text{II}7} \\
& \qquad + K_c \theta_{1\text{II}8} + \bar{K}_\rho \bar{\theta}_{1\text{II}2}]_w \\
& + O(\varepsilon^3, \varepsilon^2\bar{\varepsilon}, \varepsilon\bar{\varepsilon}^2, \bar{\varepsilon}^3).
\end{aligned} \tag{18}
$$

The leading terms in equations (17) and (18) expressing the constant property results, are of order ε and $\bar{\varepsilon}$, respectively. Since the heat transfer quantities q_w and T_w are one order of magnitude smaller than the momentum transfer quantity c_f constant property results are not those with $\varepsilon = \bar{\varepsilon} = 0$ (then $T_w = q_w = 0$) but $\varepsilon \to 0$ and $\bar{\varepsilon} \to 0$. That is why temperature results for the thermal boundary condition $T_w = \text{const}$ are usually nondimensionalized by $T_w^* - T_o^* = \varepsilon T_o^*$, a quantity of order $O(\varepsilon)$, leading to $\theta_I = (T_I^* - T_o^*)/(T_w^* - T_o^*)$ of order $O(1)$.

6. EMPIRICAL METHODS IN THE LIGHT OF THE ASYMPTOTIC RESULTS

Since compressible flow is treated like a variable property flow in this study, the question arises if empirical methods—well established in the field of variable property flows—are also applicable to compressible flows. There are two well-known empirical methods for variable property effects.

In the *property ratio method* the constant property results are multiplied by a power of some pertinent property evaluated at the surface temperature to that property at the reference temperature T_o^*.

A complete formula for the skin friction for example reads:

$$
\frac{c_f}{c_{\text{fcp}}} = \rho_w^{m_\rho} \eta_w^{m_\eta}; \quad m_\rho = \text{const}, m_\eta = \text{const}. \tag{19}
$$

In the *reference temperature method* a temperature T_r^* is specified at which the properties appearing in the dimensionless groups of the problem should be evaluated to obtain the variable property results by constant property formulae.

By means of the linear asymptotic theory the unknowns in both methods (the exponents m_ρ, m_η and the reference temperature, respectively) can be derived analytically, for details of the procedure, see ref. [7].

So a first statement is that it is a characteristic feature of these formulae that higher-order (nonlinear) effects cannot be taken into account, they are linear methods by nature.

The second statement concerns the applicability to compressible flow. In the general case there are two independent effects of variable density. One is related to $\partial\rho^*/\partial T^*$ and one to $\partial\rho^*/\partial p^*$. As long as both effects are in a problem they cannot be covered by one exponent or one reference temperature. As a consequence property ratio and reference methods are applicable only in cases where there is no effect of $\partial\rho^*/\partial p^*$, i.e. for the flat plate and stagnation point flow. An extension to general wedge type flows is not possible.

Flat plate flow

Applying the *property ratio concept* the correction formula for skin friction for example is (since ρ and η appear in a fixed combination only they can be treated like one property):

$$
\begin{aligned}
\frac{c_f}{c_{\text{fcp}}} &= (\rho_w \eta_w)^{m_{\rho\eta}} \\
&= 1 + m_{\rho\eta} K_{\rho\eta} (\varepsilon x^e \theta_{\text{oIw}} + \bar{\varepsilon} \theta_{\text{oIIw}}) + O(\varepsilon^2, \varepsilon\bar{\varepsilon}, \bar{\varepsilon}^2).
\end{aligned} \tag{20}
$$

Comparing equations (20) and (16) for $T_w = \text{const}$ leads to the following expressions for the exponent $m_{\rho\eta}$ ($\theta_{\text{oIw}} = 1$; $\theta_{\text{oIIw}} = 0$, $e = 0$):

$$
m_{\rho\eta} = m_{\rho\eta 1} + \frac{\bar{\varepsilon}}{\varepsilon} m_{\rho\eta 2} \tag{21}
$$

with

$$
m_{\rho\eta 1} = 1 + \frac{f''_{1\text{I}1w}}{f''_{\text{ow}}}
$$

$$
m_{\rho\eta 2} = \frac{f''_{1\text{II}1w}}{f''_{\text{ow}}} + \frac{\bar{K}_\rho}{K_{\rho\eta}} (\bar{f}''_{1w} - \tfrac{1}{2}) / f''_{\text{ow}}.
$$

The first part, $m_{\rho\eta 1}$, is the exponent for incompressible (low speed) flow, $m_{\rho\eta 2}$ is the deviation caused by viscous heating, the only high speed effect for a flat plate flow. The expression for $m_{\rho\eta 2}$ is simplified when the free-stream quantities T_e^*, p_e^* instead of the stagnation values T_o^*, p_o^* are taken as reference state. Then $m_{\rho\eta 2}$ is free of K-numbers and both parts, $m_{\rho\eta 1}$ and $m_{\rho\eta 2}$ are functions of the Prandtl number only, since f''_{iw} depend on Pr. For the special case of the flat plate with free-stream reference conditions their numerical values are given in ref. [7] or can be extracted from Table 2(a) by taking into account the change in the reference state properly.

From these considerations it is concluded that the free-stream values are the adequate reference state for the flat plate flow rather than stagnation values which nevertheless are used in general for uniqueness.

Applying the *reference temperature concept*, T_r^* may be written as

$$
\frac{T_r^*}{T_e^*} = a_1 + a_2 Ma_e^2 + a_3 \frac{T_w^*}{T_e^*} \quad \text{(perfect gas)}. \tag{22}
$$

In this form the reference temperature can be compared to empirical results listed in ref. [1]. The range of empirical data for $Pr = 0.7$ is:

$$
\begin{aligned}
a_1 &= 0.42\text{–}0.55 \quad (\text{exact: } a_1 = 0.468) \\
a_2 &= 0.032\text{–}0.039 \quad (\text{exact: } a_2 = 0.031) \\
a_3 &= 0.45\text{–}0.58 \quad (\text{exact: } a_3 = 0.532).
\end{aligned}
$$

The numbers of the linear asymptotic theory are listed in parentheses so that 'exact' precisely means 'exact within the linear theory'—but that is all that can be covered by the reference temperature as was mentioned before.

Stagnation point flow

For the stagnation point the general results, equations (16)–(18) reduce drastically. With $e = 0$, $x = 0$ the only higher-order terms that are left over are those of order $O(\varepsilon)$. No high speed effect expressed by $\bar{\varepsilon}$ and $\bar{K}_\rho$ is left since it is a low speed flow by nature. The only variable property effects are those of heat transfer, the correction formulae are those of low speed stagnation point flow with heat transfer. The skin friction formula after the property ratio method for example is:

$$\frac{c_f}{c_{fcp}} = (\rho_w \eta_w)^{m_{\rho\eta}} \rho_w^{m_\rho}$$
$$= 1 + m_{\rho\eta} \varepsilon K_{\rho\eta} \theta_{o1} + m_\rho \varepsilon K_\rho \theta_{o1} + O(\varepsilon^2). \tag{23}$$

Comparison with equation (16) yields:

$$m_{\rho\eta} = 1 + \frac{f''_{1I1w}}{\theta_{oIw} f''_{ow}}; \quad m_\rho = \frac{f''_{1I2w}}{\theta_{oIw} f''_{ow}}. \tag{24}$$

In contrast to equation (20) an additional factor $\rho_w^{m_\rho}$ should be introduced for a complete correction formula since ρ does not always appear in the fixed combination $\rho\eta$ as in the flat plate flow. In equation (24) either the numerical values from Table 2(a) (T_w = const, boundary condition $\theta_{oIw} = 1$) or those from Table 2(b) can be inserted since for stagnation point flow q_w = const implies T_w = const and vice versa.

By virtue of the so-called Mangler transformation, see ref. [1], the solution for $m = 1/3$ ($\beta = 1/2$) corresponds to the axisymmetric stagnation point flow. The only formal difference is that the RHS of equation (7) is multiplied by $\sqrt{2}$ in the case of axisymmetric flow with the consequence that in equation (17) the RHS must be multiplied by the same factor $\sqrt{2}$.

7. EXACT SOLUTIONS AND ASYMPTOTIC RESULTS

The deviations of the linear asymptotic results from those that take into account the variable properties completely are of order $O(\varepsilon^2, \varepsilon\bar{\varepsilon}, \bar{\varepsilon}^2)$ asymptotically. But for practical purposes the linear theory is a good approximation, even for values of ε and $\bar{\varepsilon}$ not very close to zero, as will be demonstrated by the following examples.

Zero pressure gradient

In Fig. 2 the exact solution for skin friction of a perfect gas by Van Driest [9] is compared to the linear theory of this study. Figure 2 shows a good coincidence up to high speed parameters $\bar{\varepsilon}$ of about 10

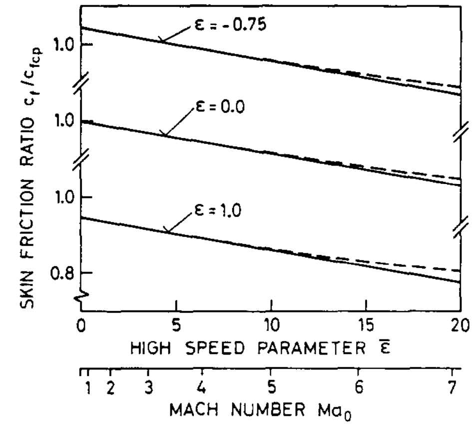

FIG. 2. Flat plate flow: T_w = const; $Pr = 0.75$; Sutherland law. --- Van Driest [9]; —— linear theory.

and heat transfer parameters of about one. In the light of an asymptotic theory for $(\varepsilon, \bar{\varepsilon}) \to 0$ this result is quite amazing. But one should keep in mind that the only high speed effect is viscous heating and that the temperature effects on ρ and η mostly compensate each other. Both properties appear only in the fixed combination $\rho\eta$, no term with ρ alone appears for $\beta = 0$ (flat plate) as can be seen in equation (9). The compensation to the first order between ρ and η is expressed by a small value for $K_{\rho\eta}$. In the example of Fig. 2 it is $K_{\rho\eta} = K_\rho + K_\eta = -1 + 0.788 = -0.212$.

An interesting feature of adiabatic flow is the adiabatic wall temperature T^*_{aw} and the recovery factor r defined by

$$r = \frac{T^*_{aw} - T^*_e}{u_e^{*2}/2c^*_{pe}}. \tag{25}$$

For the adiabatic case q_w is zero, i.e. $\varepsilon = 0$. The dimensionless temperature $\bar{\varepsilon}\theta_{II}$ is then $(T^*_{aw} - T^*_o)/T^*_o$ so that the recovery factor reads ($u^*_e = u^*_o$ for the flat plate):

$$r \equiv 2\frac{T^*_{aw} - T^*_e}{u_e^{*2}/c^*_{pe}} = \frac{2}{\bar{\varepsilon}}\left(\frac{T^*_{aw} - T^*_o}{T^*_o} + \frac{T^*_o - T^*_e}{T^*_o}\right)\frac{c^*_{pe}}{c^*_{po}}$$
$$= (1 + 2\theta_{oII}) + 2\bar{\varepsilon}\left[K_{\rho\eta}\theta_{1II5} + K_{\rho\lambda}\theta_{1II7} + K_c\left(\theta_{1II8} - \frac{\theta_{oII}}{2} - \frac{1}{8}\right) + \bar{K}_\rho \bar{\theta}_{1II2}\right] + O(\bar{\varepsilon}^2). \tag{26}$$

For $Pr = 1$ for example, with the numerical values from Table 2(b), the well-known result $r = 1$ follows which even holds for compressible flow as far as the zero-order results are concerned. The first-order deviations from this constant are weak since all θ_{1IIi} are small numbers in the vicinity of $Pr = 1$ as can be seen in Table 2(b).

Non-zero pressure gradient

For the special case of a perfect gas with the Sutherland viscosity law and for $Pr = 1$, Cohen and Reshotko [10] treated the problem by a

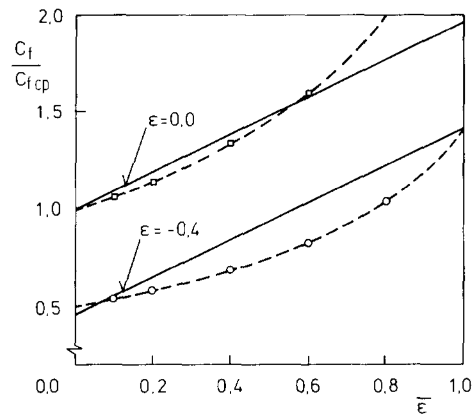

FIG. 3(a). Falkner-Skan: $m = 1/3$; $T_w = \text{const}$; $Pr = 1.0$; $x = 1.0$. ○□ Cohen and Reshotko [10]; —— linear theory.

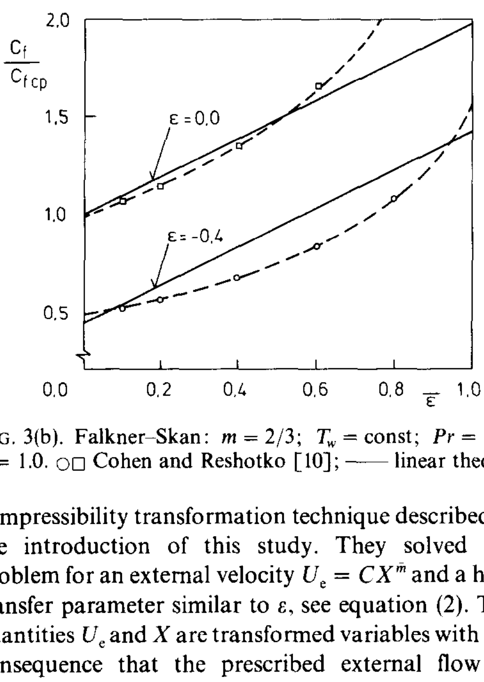

FIG. 3(b). Falkner-Skan: $m = 2/3$; $T_w = \text{const}$; $Pr = 1.0$; $x = 1.0$. ○□ Cohen and Reshotko [10]; —— linear theory.

compressibility transformation technique described in the introduction of this study. They solved the problem for an external velocity $U_e = CX^{\bar{m}}$ and a heat transfer parameter similar to ε, see equation (2). The quantities U_e and X are transformed variables with the consequence that the prescribed external flow in original physical variables is of power law type only for $Ma = 0$. The Mach number dependence of the problem is hidden in the transformation and not explicit as in the asymptotic approach.

To compare the results from ref. [10] with the power law external velocity results of this study, a local correspondence between the power $\bar{m}$ from ref. [10] and m according to equation (6) is assumed. This again is a non-asymptotic approximation which is necessary to compare the results of the two theories. For the two cases, $m = 1/3$ and $m = 2/3$ and a fixed location $x = 1$ (all other values are possible), the two theories are compared in Figs. 3(a) and (b). There is a satisfactory coincidence for rather large values of the heat transfer parameter ε as well as for the high speed parameter $\bar{\varepsilon}$.

8. DISCUSSION

There are three important features of the asymptotic approach to compressible flow that should be emphasized:

(1) The typical advantage of a perturbation technique holds: the results are general in the sense that a specification to certain flow cases is made in the results only (by specifying $\varepsilon, \bar{\varepsilon}$ and the fluid through $K_\rho, K_\eta, \ldots$).

(2) An additional advantage is that the influence of the physical properties can be checked separately. This statement applies especially to 'compressibility effects' (associated with $\bar{\varepsilon}$ and $\bar{K}_\rho$, respectively) in contrast to the alternative method of hiding it in compressibility transformations.

(3) As far as laminar flow is concerned all information is extracted from the basic equations. Based on these results well-known empirical methods to account for variable property effects can be understood as theoretical methods (see Section 6).

Finally the question may be answered that really was the starting point for this study: what is a compressible flow?

The most general answer is: a flow with non-constant density, i.e. variations either through $\partial\rho/\partial p$ or $\partial\rho/\partial T$ are involved in the problem. Due to this definition the flat plate flow at $Ma \neq 0$ and stagnation point flow with heat transfer are compressible flows since variations in density are present through $d\rho/dT$. The same argument holds for every flow with $Ma = 0$ but non-zero heat transfer. It is suggested that this definition of compressibility be called *compressibility in a general sense.*

A more restrictive answer is: a flow with non-constant density through the effect of $\partial\rho/\partial p$. A necessary condition for this is high speed, i.e. $Ma \neq 0$. This definition may be called *compressibility in an aerodynamic sense*, since it is typical for aerodynamic high speed flow situations. But it should be kept in mind that high speed is necessary but not sufficient. The boundary layer at a flat plate, even at supersonic Mach numbers, is incompressible in that sense since the pressure is constant in this case and variations in density occur through $\partial\rho/\partial T$ (viscous heating) only.

Acknowledgements—The author would like to thank Dipl.-Ing. F.-J. Bauhaus and Prof. Dr.-Ing. K. Gersten for the intensive discussions and many helpful comments.

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APPENDIX

After inserting equations (12)–(14) and the property expansions into the basic equations (9) and (10) two sets of ordinary differential equations are obtained which are the zero- and first-order equations with respect to ε and $\bar{\varepsilon}$, respectively.

For convenience the following differential operators are introduced:

$$L_1(f) = f''' + f_o f'' - \left(2\beta + \frac{2e}{m+1}\right) f_o' f' + \left(1 + \frac{2e}{m+1}\right) f_o'' f \tag{A1}$$

$$L_2(f) = f''' + f_o f'' - 4\beta f_o' f' + (2\beta+1) f_o'' f \tag{A2}$$

$$L_3(f,H) = \frac{H''}{Pr_o} + f_o' H - \frac{4e}{m+1} f_o' H - \frac{2e}{m+1} f' H_{oI} + \left(1 + \frac{2e}{m+1}\right) f H_{oI}' \tag{A3}$$

$$L_4(f,H) = \frac{H''}{Pr_o} + f_o' H' - 4\beta f_o' H - 2\beta f' H_{oII} + (2\beta+1) f H_{oII}' - \left(\frac{1}{Pr_o} - 1\right)(f' f_o')'' \tag{A4}$$

$$L_5(f,g,H) = \frac{H''}{Pr_o} + f_o H' - \left(2\beta + \frac{2e}{m+1}\right) f_o' H - 2\beta f' H_{oII} - \frac{2e}{m+1} g' H_{oI} + \left(\frac{2e}{m+1} + 1\right) f H_{oII}' + (2\beta+1) g H_{oI}' - \left(\frac{1}{Pr_o} - 1\right)(f' f_o')''. \tag{A5}$$

With L_1–L_5 the momentum and energy equations are:

zero order with respect to ε and $\bar{\varepsilon}$:

$$f_o''' + f_o f_o'' + \beta(1 - f_o'^2) = 0 \tag{A6}$$

$$\frac{H_{oI}''}{Pr_o} + f_o H_{oI}' - \frac{2e}{m+1} f_o' H_{oI} = 0 \tag{A7}$$

$$\frac{H_{oII}''}{Pr_o} + f_o H_{oII}' - 2\beta f_o' H_{oII} = \left(\frac{1}{Pr_o} - 1\right)(f_o' f_o'')' \tag{A8}$$

with the associated boundary conditions [(α): $T_w = \text{const}$, (β): $q_w = \text{const}$]:

$$\eta_s = 0:\quad f_o = f_o' = 0;\ (\alpha): H_{oI} - 1 = H_{oII} = 0;\quad (\beta): H_{oI}' + 1 = H_{oII}' = 0 \tag{A9}$$

$$\eta_s \to \infty:\quad f_o' - 1 = H_{oI} = H_{oII} = 0 \tag{A10}$$

first order with respect to ε and $\bar{\varepsilon}$:

$$\left.\begin{aligned}
&L_1(f_{1\mathrm{I}1}) = -(\theta_{o\mathrm{I}} f_o'')' &&\text{(A11)}\\
&L_2(f_{1\mathrm{II}1}) = -(\theta_{o\mathrm{II}} f_o'')' &&\text{(A12)}\\
&L_3(f_{1\mathrm{I}1}, H_{1\mathrm{I}1}) = 0 &&\text{(A13)}\\
&L_4(f_{1\mathrm{II}1}, H_{1\mathrm{II}5}) = -(\theta_{o\mathrm{II}} f_o' f_o'')' &&\text{(A14)}\\
&L_5(f_{1\mathrm{I}1}, f_{1\mathrm{II}1}, H_{1\mathrm{II}1}) = -(\theta_{o\mathrm{I}} f_o' f_o'')' &&\text{(A15)}
\end{aligned}\right\}\ \text{associated with } K_{\rho\eta}$$

$$\left.\begin{aligned}
&L_1(f_{1\mathrm{I}2}) = \beta\theta_{o\mathrm{I}} &&\text{(A16)}\\
&L_2(f_{1\mathrm{II}2}) = \beta(\theta_{o\mathrm{II}} + 1/2) &&\text{(A17)}\\
&L_3(f_{1\mathrm{I}2}, H_{1\mathrm{I}2}) = 0 &&\text{(A18)}\\
&L_4(f_{1\mathrm{II}2}, H_{1\mathrm{II}6}) = 0 &&\text{(A19)}\\
&L_5(f_{1\mathrm{I}2}, f_{1\mathrm{II}2}, H_{1\mathrm{II}2}) = 0 &&\text{(A20)}
\end{aligned}\right\}\ \text{associated with } K_{\rho}$$

$$\left.\begin{aligned}
&L_3(0, H_{1\mathrm{I}3}) = -Pr_o^{-1}(\theta_{o\mathrm{I}}\theta_{o\mathrm{I}}')' &&\text{(A21)}\\
&L_4(0, H_{1\mathrm{II}7}) = -Pr_o^{-1}(\theta_{o\mathrm{II}}\theta_{o\mathrm{II}}')' &&\text{(A22)}\\
&L_5(0, 0, H_{1\mathrm{II}3}) = -Pr_o^{-1}(\theta_{o\mathrm{I}}\theta_{o\mathrm{II}})'' &&\text{(A23)}
\end{aligned}\right\}\ \text{associated with } K_{\rho\lambda}$$

$$\left.\begin{aligned}
&L_3(0, H_{1\mathrm{I}4}) = Pr_o^{-1}(\theta_{o\mathrm{I}}\theta_{o\mathrm{I}}')' &&\text{(A24)}\\
&L_4(0, H_{1\mathrm{II}8}) = Pr_o^{-1}(\theta_{o\mathrm{II}}\theta_{o\mathrm{II}}')' &&\text{(A25)}\\
&L_5(0, 0, H_{1\mathrm{II}4}) = Pr_o^{-1}(\theta_{o\mathrm{I}}\theta_{o\mathrm{II}})'' &&\text{(A26)}
\end{aligned}\right\}\ \text{associated with } K_{c}$$

$$\left.\begin{aligned}
&L_2(\bar{f}_1) = f_o'''/2 &&\text{(A27)}\\
&L_4(\bar{f}_1, \bar{H}_{1\mathrm{II}2}) = Pr_o^{-1}\theta_{o\mathrm{II}}''/2 + (f_o' f_o'')'/2 &&\text{(A28)}\\
&L_5(0, \bar{f}_1, \bar{H}_{1\mathrm{II}1}) = Pr_o^{-1}\theta_{o\mathrm{I}}''/2 &&\text{(A29)}
\end{aligned}\right\}\ \text{associated with } \bar{K}_{\rho}$$

with the associated boundary conditions:

$$\begin{aligned}
\eta_s = 0:\quad & f_i = \bar{f}_i = f_i' = \bar{f}_i' = 0 \quad (\text{all } i)\\
& (\alpha): H_{1\mathrm{I}i} = H_{1\mathrm{II}j} = \bar{H}_{1\mathrm{II}k} = 0 \qquad (i = 1\text{–}3;\ j = 1\text{–}8;\ k = 1,2)\\
& H_{1\mathrm{I}4} = 1/2\\
& (\beta): H_{1\mathrm{I}i}' = H_{1\mathrm{II}j}' = \bar{H}_{1\mathrm{II}2}' = 0 \qquad (i = 1,2;\ j = 1,2,5\text{–}8)\\
& H_{1\mathrm{I}3}' = -H_{1\mathrm{I}4}' = H_{o\mathrm{I}};\\
& H_{1\mathrm{II}3}' = -H_{1\mathrm{II}4}' = H_{o\mathrm{II}};\ H_{1\mathrm{II}1}' = -1/2
\end{aligned} \tag{A30}$$

$$\eta_s \to \infty:\ f_i' = \bar{f}_i' = H_i = \bar{H}_i = 0 \quad (\text{all } i). \tag{A31}$$

The first-order relations between total enthalpy and temperature are:

$$\theta_{1\mathrm{I}i} = H_{1\mathrm{I}i};\quad i = 1\text{–}3 \tag{A32}$$

$$\theta_{1\mathrm{I}4} = H_{1\mathrm{I}4} - \theta_{o\mathrm{I}}^2/2 \tag{A33}$$

$$\dots$$

$$\theta_{1\mathrm{II}i} = H_{1\mathrm{II}i} - f_o' f_{1\mathrm{I}i}';\quad i = 1,2 \tag{A34}$$

$$\theta_{1\mathrm{II}3} = H_{1\mathrm{II}3} \tag{A35}$$

$$\theta_{1\mathrm{II}4} = H_{1\mathrm{II}4} - \theta_{o\mathrm{I}}\theta_{o\mathrm{II}} \tag{A36}$$

$$\theta_{1\mathrm{II}5} = H_{1\mathrm{II}5} - f_o' f_{1\mathrm{II}1}' \tag{A37}$$

$$\theta_{1\mathrm{II}6} = H_{1\mathrm{II}6} - f_o' f_{1\mathrm{II}2}' \tag{A38}$$

$$\theta_{1\mathrm{II}7} = H_{1\mathrm{II}7} \tag{A39}$$

$$\theta_{1\mathrm{II}8} = H_{1\mathrm{II}8} - \theta_{o\mathrm{II}}^2/2 \tag{A40}$$

$$\bar{\theta}_{1\mathrm{II}1} = \bar{H}_{1\mathrm{II}1} \tag{A41}$$

$$\bar{\theta}_{1\mathrm{II}2} = \bar{H}_{1\mathrm{II}2} - f_o' \bar{f}_1'. \tag{A42}$$

UNE APPROCHE ASYMPTOTIQUE DE L'ECOULEMENT COMPRESSIBLE DE COUCHE LIMITE

Résumé—Une approche par perturbation est appliquée pour tenir compte de la variation des propriétés. On s'intéresse spécialement à l'effet de la densité variable avec la pression. Dans le cadre d'une théorie asymptotique, les effets de compressibilité sont considérés comme des effets de propriétés variables. A l'aide de cette théorie asymptotique, la déviation du frottement pariétal et du transfert de chaleur relativement aux valeurs du cas isotherme et incompressible est déterminée pour les couches limites laminaires de Falkner–Skan. Tant que l'on considère l'écoulement laminaire, il n'y a pas besoin d'information empirique.

EINE ASYMPTOTISCHE THEORIE FÜR KOMPRESSIBLE GRENZSCHICHTSTRÖMUNGEN

Zusammenfassung—Zur Erfassung des Einflusses variabler Stoffwerte wird eine reguläre Störungsrechnung durchgeführt. Von besonderem Interesse ist dabei die Druckabhängigkeit der Dichte. Im Rahmen einer asymptotischen Theorie werden Kompressibilitätseffekte als variable Stoffwert-Effekte betrachtet. Mit Hilfe der asymptotischen Theorie werden die Abweichungen der Schubspannungs- und Wärmeübertragungsergebnisse von ihren inkompressiblen, isothermen Werten für laminare Falkner–Skan Strömungen hergeleitet. Für laminare Strömungen bedarf es dafür keinerlei emprischer Information.

ИСПОЛЬЗОВАНИЕ АСИМПТОТИЧЕСКОГО МЕТОДА ОПИСАНИЯ ТЕЧЕНИЯ СЖИМАЕМОГО ПОГРАНИЧНОГО СЛОЯ

Аннотация—Для учета эффектов переменности свойств использован метод регулярных возмущений. Особый интерес представляет зависимость плотности от давления. В рамках асимптотической теории влияние сжимаемости рассматривается как влияние переменности свойств. С помощью асимптотической теории определяется отклонение коэффициента поверхностного трения и характеристик теплопереноса от соответствующих значений для несжимаемого изотермического ламинарного пограничного слоя Фолкнера–Скэна. Поскольку рассматривается ламинарное течение, то не возникает необходимость использовать эмпирические соотношения.